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Direct Passive Navigation: Analytical Solution for Quadratic Patches

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Abstract: In this paper, we solve the problem of recovering the motion of an observer relative to a surface which can be locally approximated by a quadratic patch directly from image brightness values. We do not compute the optical flow as an intermediate step. We use the coefficients of the Taylor series expansion of the intensity function in two frames to determine 15 intermediate parameters, termed the *essential parameters*, from a set of linear equations. We then solve analytically for the motion and structure parameters from a set of nonlinear equations in terms of these intermediate parameters. We show that the solution is always unique, unlike some earlier results that reported two-fold ambiguities in some special cases.

Key Words: Passive Navigation, Motion Field, Optical Flow, Structure and Motion, Quadratic Patches, Taylor Series Expansion, Least-Square-Errors, Essential Parameters.

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1. Introduction

The problem of recoverying rigid body motion and surface structure uniquely from image data has been the topic of many research papers in the area of machine vision [1-25]. Many approaches based on matching feature points [5,11,21,22] tracking contours [9], and using velocity flow field [1,3,4,10,12,18,19,23-25], texture [2], or image intensity gradients [14-17] have been proposed in the literature, but not many have addressed the important issue of uniqueness.

Feature point matching schemes require the detection of local brightness patterns that are likely to be found in consecutive images. A correspondence problem between the elements in successive 2-D images is first established and then the 3-D motion and spatial configuration of these isolated features is recovered. The minimum number of points required to recover the 3-D motion uniquely depends on the number of image frames. With 2 frames, in most cases, a minimum of 5 points results in a unique solution from a set of nonlinear equations. Using 8 points, however, as in algorithms proposed by Longuet-Higgins [11], Tsai and Huang [22], Buxton et al. [5], one only solves linear equations. In any case, these methods require feature detection and matching that fail to give reliable results when the object in view is smooth with no well-defined features. Further, since these methods use information only from a small portion of the image, they are noise sensitive. When nearby feature points are selected, these methods become even more sensitive to small amounts of error in the data.

For smooth curved surfaces, Longuet-Higgins and Prazdny [10] suggested a method that uses the optical flow and its first and second derivatives at a single point. They reduced the problem to that of solving a cubic equation and concluded that, in general, three solutions are feasible. Later, Waxman and Ullman [23] developed the method proposed in [11] into an algorithm for recovering the structure and motion parameters from a set of nonlinear equations. They had to treat many special cases, and uniqueness results were shown through numerical examples. Waxman et al. [25] found a closed form solution to the original formulation in [23]. Apparently however, in their formulation, they are not able to resolve the two-fold ambiguities that arise in two special cases. When the observer translates parallel to the line of sight they cannot recover the curvature parameters. The result is a two-fold ambiguity similar to the one associated with planar surfaces. Also when there is no (component of) motion in the direction of zero surface gradient (labeled as *structure motion coincidence* in Waxman and Ullman [23]), they cannot distinguish between the two possible solutions. In any case, these methods are very noise sensitive since second order derivatives of errorful optical flow data are used. More robust algorithms that use the information from the whole region of the image plane have been suggested [1,3,4], but one still has to compute a dense flow field from a sequence of images.

The flow field based approaches assume that a reasonable estimate of the optical flow is available. In general, the computation of the local velocity field exploits a constraint

equation between the local intensity changes and the two components of the optical flow. This only gives the component of the flow in the direction of the intensity gradient. To compute the full flow field, one needs additional constraints such as the heuristic assumption that the flow field is locally smooth [7,8]. In many cases, this leads to optical flow fields that are not consistent with the true velocity field. Since, in general, velocity based approaches are not robust, i.e., the solution may change drastically with only a small amount of noise in the data, one can argue that approaches that use an optical flow field that is computed through heuristic assumptions are apt to fail.

In this paper, we present a method that uses the intensity values in a sequence of images directly, in order to recover the motion parameters as well as the local structure of the surface patches on the object. Since we do not compute the optical flow, we do not need to make any heuristic assumptions. We assume that the surface is textured or has surface markings, and that it is smooth so that we can approximate it in a local region by a quadratic patch. We give a closed form solution for the motion and surface parameters, and show that the solution is always unique.

In a paper in press, we have extended our methodology presented in this paper to recover the motion of any smooth surface that can be presented by an n^{th} order Taylor series expansion (with no restriction on n) uniquely. We show uniqueness and give closed form solution for the motion and surface parameters.

2. Preliminaries

We first recall some details about perspective projection, rigid body motion, the motion field and optical flow, and the brightness change constraint equation. This we do using the tensor notation (summation over repeated indices) and the following convention. A 2-D vector $\mathbf{v}$ will be referred to by v_i, and a 3-D vector $\boldsymbol{\omega}$ by $\{w_i, w_3\}$, where w_i is the component of $\boldsymbol{\omega}$ parallel to the image plane.

2.1. Perspective Projection

Let the center of projection be at the origin of a Cartesian coordinate system. Without loss of generality we assume that the effective focal length is unity. The image is formed on the plane $z = 1$. Therefore, the optical axis lies along the z-axis. Let $\{Xi, Z\}$ be a point in the scene. Its projection in the image is $\{x_i\}$, where:

$$x_i = \frac{X_i}{Z}.$$

2.2. Rigid Body Motion

In the case of the observer moving relative to a rigid environment with translational velocity $\{t_i, t_3\}$ and rotational velocity $\{w_i, w_3\}$, we find that the motion of a point in

the environment relative to the observer is given by:

$$\frac{dX_i}{dt} = -\epsilon_{ij}(Zw_j - X_j w_3) - t_i, \qquad \frac{dZ}{dt} = \epsilon_{ij} X_i w_j - t_3,$$

where ϵ_{ij}, the anti-symmetric permutation tensor, is given by:

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2.3. Image Motion Field and Optical Flow

The projection of the 3-D velocity field, induced by a rigid body motion, onto the image plane is known as the *image motion field.* It is a purely geometric concept that is uniquely defined when the surface and the observer motion are given. It can be determined by differentiating the perspective projection relationship $(X_i = x_i Z)$ with respect to time:

$$\frac{dx_i}{dt} = \frac{1}{Z}\left(\frac{dX_i}{dt} - x_i \frac{dZ}{dt}\right).$$

Substituting for dX_i/dt and dZ/dt, and simplifying the results, we obtain:

$$\frac{dx_i}{dt} = -\epsilon_{kj}(\delta_{ik} + x_i x_k) w_j + x_j \epsilon_{ij} w_3 + \frac{1}{Z}(x_i t_3 - t_i),$$

where δ_{ij} is the Kronecker delta symbol (it equals 1 when $i = j$ and zero otherwise). It is important to remember that there is an inherent ambiguity here, since the same motion field results when distance and the translational velocity are multiplied by an arbitrary constant. This can be seen easily from the above equation since the same image plane velocity is obtained if one multiplies both Z and $\{t_i, t_3\}$ by some constant. So we conclude that the surface structure and the translation motion parameters can be recovered only up to a scale factor.

The *optical flow* is the apparent motion of the brightness patterns. An optical flow field is a vector field that shows how the brightness patterns at one instance of time can be transformed into that at the next instance of time. It is something we hope to estimate from an image sequence, but it is not uniquely defined. Additional assumptions must be made to guarantee a unique result.

The distinction between the motion field and the optical flow is an important one that is rarely made. To recover motion, we need the motion field, but the best we can do is to estimate an optical flow field from an image sequence. In extreme cases, the two will be very different: consider, for example, a perfectly smooth and uniform rotating billiard ball, or the motion of the shadows cast on a stationary object due to a moving light source. Under favorable circumstances the optical flow is identical to the motion field. Here, we assume that the motion flow field and the optical flow are the same.

2.4. Brightness Change Constraint Equation

The brightness of the image of a particular patch of a surface depends on many factors. It may for example vary with the orientation of the patch. In many cases, however, it remains at least approximately constant as the surface moves in the environment. This is a good assumption when the surface has strong texture or surface markings, and is commonly used in work on optical flow. If we assume that the image brightness of a patch is indeed constant, we have

$$\frac{dE}{dt} = 0,$$

or, using the chain rule:

$$\frac{\partial E}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial E}{\partial t} = 0,$$

where $E_{x_i} = \partial E/\partial x_i$, and $E_t = \partial E/\partial t$ are the spacial and temporal derivatives of the the image brightness function. Substituting for dx_i/dt we get:

$$E_t - (\delta_{ik} + x_i x_k)\epsilon_{kj} E_{x_i} w_j + x_j \epsilon_{ij} E_{x_i} w_3 + \frac{1}{Z}(x_i E_{x_i} t_3 - E_{x_i} t_i) = 0.$$

If we let $d = 1/Z$, and define:

$$S_i = -E_{x_i}, \qquad S_3 = x_i E_{x_i},$$

$$V_i = \epsilon_{ij}(x_j S_3 - S_j), \qquad V_3 = -\epsilon_{ij} S_i x_j,$$

then:

$$E_t + V_i w_i + V_3 w_3 + d(S_i t_i + S_3 t_3) = 0.$$

This is the ***brightness change constraint equation*** in the case of rigid body motion.

3. Surface Representation

In a small neighborhood, a smooth surface $Z(X_i)$ can be represented by its Taylor series expansion about a reference point. Without loss of generality, let us consider a neighborhood of the line of sight, denoted by Γ_z, and choose the point of fixation, O, located at a distance Z_0 from the viewer as the reference point. Clearly, the origin of the image plane is the image of the point of fixation. Then we can write:

$$Z(X_i) = Z_0 + (\frac{\partial Z}{\partial X_i})_0 X_i + \frac{1}{2}(\frac{\partial^2 Z}{\partial X_i \partial X_j})_0 X_i X_j + \ldots.$$

If we neglect the higher order terms, we are left with a quadratic approximation to the surface (see Figure 1a). While this representation is globally valid for quadratic surfaces, it may only be locally valid for other smooth surfaces. We can express the series expansion in terms of the image coordinates, x_i, by employing the perspective projection

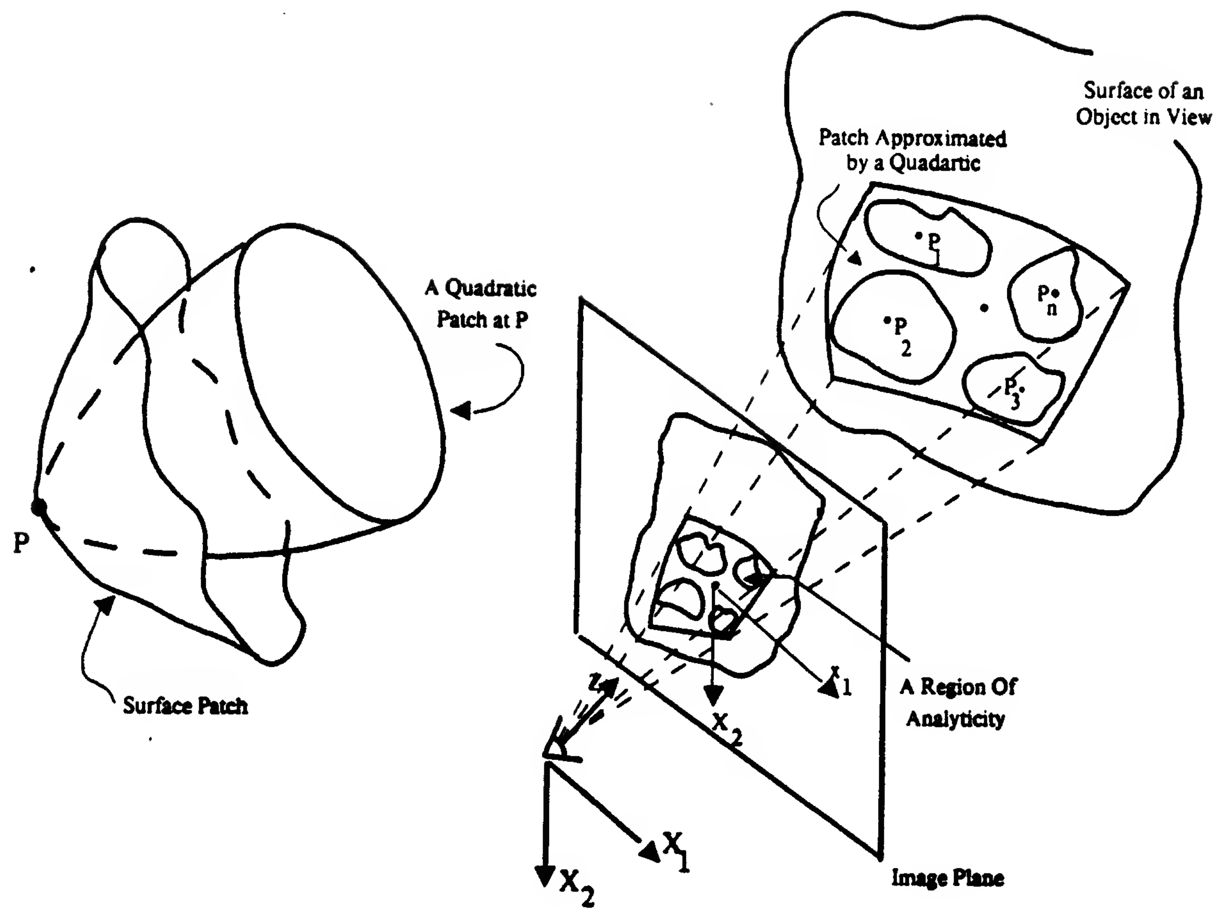

Figure 1: (a) A Quadratic Approximation to the Surface Patch at P
(b) Image of a Surface Patch and Regions of Analyticity

relationship ($X_i = x_i Z$). We consider the series expansion of $d = 1/Z$ instead, since the brightness change equation is in terms of d. The surface is then represented by:

$$d(x_i) = d_0 + (\frac{\partial d}{\partial x_i})_0 x_i + \frac{1}{2}(\frac{\partial^2 d}{\partial x_i \partial x_j})_0 x_i x_j + ...,$$

where it can be shown (see appendix 3) that:

$$(\frac{\partial d}{\partial x_i})_0 = -\frac{1}{Z_0}(\frac{\partial Z}{\partial X_i})_0, \quad (\frac{\partial^2 d}{\partial x_i \partial x_j})_0 = -(\frac{\partial^2 Z}{\partial X_i \partial X_j})_0.$$

Since the surface parameters and the translation motion can only be recovered up to a scale factor, we can arbitrarily fix one of the coefficients in the above series expansion. We choose to set $d_0 = 1/Z_0 = 1$. Therefore:

$$d(x_i) = 1 + d_i x_i + \frac{1}{2} d_{ij} x_i x_j + ...,$$

where

$$d_i = (\frac{\partial d}{\partial x_i})_0, \qquad d_{ij} = (\frac{\partial^2 d}{\partial x_i \partial x_j})_0.$$

In this paper, we will assume that the above series expansion, up to second order terms, is a good approximation to $1/Z$ in Γ_z.

4. Essential Parameters for a Quadratic Patch

Let A_z denote the image of the surface patch Γ_z. We assume that the patch has texture or surface markings. More precisely, we assume that in small regions in A_z corresponding to areas of texture, the intensity variation is smooth so that locally, we can approximate the intensity function by a Taylor series expansion. Equivalently, the image intensity function is assumed analytic in small areas that we refer to as *regions of analyticity.* A region of analyticity is contained within its *boundary of analyticity* (a similar term has been used by Waxman and Ullman [23] in a different context). This is illustrated in Figure 1a, where each bounded area in the image represents a region of analyticity.

Now, consider the Taylor series expansion of the image intensity function, E, and its temporal derivative, E_t, about a point p in an analytic region A_p:

$$E = \widetilde{E}_0 + \widetilde{E}_i \delta x_i + \frac{1}{2}\widetilde{E}_{ij}\delta x_i \delta x_j + \frac{1}{6}\widetilde{E}_{ijk}\delta x_i \delta x_j \delta x_k + \frac{1}{24}\widetilde{E}_{ijkl}\delta x_i \delta x_j \delta x_k \delta x_l + ...,$$

$$E_t = \widetilde{T}_0 + \widetilde{T}_i \delta x_i + \frac{1}{2}\widetilde{T}_{ij}\delta x_i \delta x_j + \frac{1}{6}\widetilde{T}_{ijk}\delta x_i \delta x_j \delta x_k + \frac{1}{24}\widetilde{T}_{ijkl}\delta x_i \delta x_j \delta x_k \delta x_l + ...,$$

where

$$\delta x_i = x_i - x_i^p.$$

If we define:

$$E_0 = \widetilde{E}_0 - \widetilde{E}_i x_i^p + \frac{1}{2}\widetilde{E}_{ij} x_i^p x_j^p - \frac{1}{6}\widetilde{E}_{ijk} x_i^p x_j^p x_k^p + \frac{1}{24}\widetilde{E}_{ijkl} x_i^p x_j^p x_k^p x_l^p - ...,$$

$$E_i = \widetilde{E}_i - \widetilde{E}_{ij} x_j^p + \frac{1}{2}\widetilde{E}_{ijk} x_j^p x_k^p - \frac{1}{6}\widetilde{E}_{ijkl} x_j^p x_k^p x_l^p + ...,$$

$$E_{ij} = \widetilde{E}_{ij} - \widetilde{E}_{ijk} x_k^p + \frac{1}{2}\widetilde{E}_{ijkl} x_k^p x_l^p - ..., \quad E_{ijk} = \widetilde{E}_{ijk} - \widetilde{E}_{ijkl} x_l^p + ..., \quad E_{ijkl} = \widetilde{E}_{ijkl} - ....,$$

$$T_0 = \widetilde{T}_0 - \widetilde{T}_i x_i^p + \frac{1}{2}\widetilde{T}_{ij} x_i^p x_j^p - \frac{1}{6}\widetilde{T}_{ijk} x_i^p x_j^p x_k^p + \frac{1}{24}\widetilde{T}_{ijkl} x_i^p x_j^p x_k^p x_l^p - ...,$$

$$T_i = \widetilde{T}_i - \widetilde{T}_{ij} x_j^p + \frac{1}{2}\widetilde{T}_{ijk} x_j^p x_k^p - \frac{1}{6}\widetilde{T}_{ijkl} x_j^p x_k^p x_l^p + ...,$$

$$T_{ij} = \widetilde{T}_{ij} - \widetilde{T}_{ijk} x_k^p + \frac{1}{2}\widetilde{T}_{ijkl} x_k^p x_l^p - ..., \quad T_{ijk} = \widetilde{T}_{ijk} - \widetilde{T}_{ijkl} x_l^p + ..., \quad T_{ijkl} = \widetilde{T}_{ijkl} - ....,$$

then for the neighborhood A_p, we can express the Taylor series expansion of the intensity function and its temporal derivative, about the origin as:

$$E = E_0 + E_i x_i + \frac{1}{2} E_{ij} x_i x_j + \frac{1}{6} E_{ijk} x_i x_j x_k + \frac{1}{24} E_{ijkl} x_i x_j x_k x_l + ...,$$

$$E_t = T_0 + T_i x_i + \frac{1}{2} T_{ij} x_i x_j + \frac{1}{6} T_{ijk} x_i x_j x_k + \frac{1}{24} T_{ijkl} x_i x_j x_k x_l + ...$$

Each region of analyticity will give rise to an independent Taylor series expansion for the intensity function and its temporal derivative. We will show that if the patch is strongly textured such that higher order coefficients of the intensity function, as in the above series expansions, exist and can be computed realiably, then we can compute the appropriate parameters from the information in only one region of analyticity. For reasons that become apparent, if we restrict ourselves to second order terms of the Taylor series expansion, then we need at least two such regions to recover the surface and motion parameters. In most cases, the texture varies more rapidly than the surface, and therefore, several such regions exist within each quadratic surface patch. This is a reasonable assumption for most textured surfaces. Similar assumptions are generally used to determine structure from motion. When several regions exist, then a least-squares formulation can be implemented to exploit the information from every region.

The spatial image intensity gradient is given by:

$$E_{x_i} = E_i + E_{ij} x_j + \frac{1}{2} E_{ijk} x_j x_k + \frac{1}{6} E_{ijkl} x_j x_k x_l + \frac{1}{24} E_{ijklm} x_j x_k x_l x_m + ... \ .$$

Using the expression for the image intensity derivatives in the definition of $\{S_i, S_3\}$ and $\{V_i, V_3\}$ we arrive at:

$$S_i = -(E_i + E_{ij} x_j + \frac{1}{2} E_{ijk} x_j x_k + \frac{1}{6} E_{ijkl} x_j x_k x_l + \frac{1}{24} E_{ijklm} x_j x_k x_l x_m + ...),$$

$$S_3 = E_i x_i + E_{ij} x_i x_j + \frac{1}{2} E_{ijk} x_i x_j x_k + \frac{1}{6} E_{ijkl} x_i x_j x_k x_l + ...,$$

$$\begin{aligned} V_i = {} & \epsilon_{ij} E_j + \epsilon_{ij} E_{jk} x_k + \epsilon_{ij} E_k x_j x_k + \frac{1}{2} \epsilon_{ij} E_{jkl} x_k x_l + \epsilon_{ij} E_{kl} x_j x_k x_l + \\ & \frac{1}{6} \epsilon_{ij} E_{jklm} x_k x_l x_m + \frac{1}{2} \epsilon_{ij} E_{klm} x_j x_k x_l x_m + \frac{1}{24} \epsilon_{ij} E_{jklmn} x_k x_l x_m x_n + ..., \end{aligned}$$

$$V_3 = -(\epsilon_{ij} E_j x_i + \epsilon_{ij} E_{jk} x_i x_k + \frac{1}{2} \epsilon_{ij} E_{jkl} x_i x_k x_l + \frac{1}{6} \epsilon_{ij} E_{jklm} x_i x_k x_l x_m + ...).$$

Substituting for $\{S_i, S_3\}$, $\{V_i, V_3\}$, d, and E_t in the brightness change constraint equation, and ignoring higher order terms, we get:

$$p_0 + p_i x_i + \frac{1}{2} p_{ij} x_i x_j + \frac{1}{6} p_{ijk} x_i x_j x_k + \frac{1}{24} p_{ijkl} x_i x_j x_k x_l = 0,$$

where

$$p_{0} = T_{0} + m_j E_j,$$

$$p_i = T_i + m_{ij} E_j + m_j E_{ji},$$

$$p_{ij} = T_{ij} + m_{ijk} E_k + 2m_{ik} E_{kj} + m_k E_{kij},$$

$$p_{ijk} = T_{ijk} + \widetilde{m}_{ij} E_k + 3m_{ljk} E_{li} + 3m_{lk} E_{lij} + m_l E_{lijk},$$

$$p_{ijkl} = T_{ijkl} + 6\widetilde{m}_{ij} E_{kl} + 6m_{mij} E_{mkl} + 4m_{mi} E_{mjkl} + m_m E_{mijkl},$$

and

$$m_i = \epsilon_{ji} w_j - t_i,$$

$$m_{ij} = \delta_{ij} t_3 - \epsilon_{ij} w_3 - d_i t_j, \qquad \widetilde{m}_{ij} = t_3 d_{ij},$$

$$m_{ijk} = (\epsilon_{lj} w_l + d_j t_3)\delta_{ki} + (\epsilon_{ki} w_k + d_i t_3)\delta_{kj} - t_k d_{ij}.$$

This is the Taylor series expansion of the brightness change constraint equation for the motion of an observer with respect to a locally quadratic surface. We will term m_i, m_{ij}, $\widetilde{m}_{ij}$, and m_{ijk} the ***essential parameters*** (these are similar to the ones we defined in an earlier paper [16] for the case of the motion of an observer with respect to a planar surface, in agreement with Tsai and Huang [21]) since as we see later, these are essential to recovering the motion and surface parameters uniquely. We will first compute these parameters using the image intensity gradients in local neighborhoods, and then use them to recover the unknowns ($\{t_i, t_3\}$,$\{w_i, w_3\}$, and $\{d_i, d_{ij}\}$).

We should remind the reader that if we do not restrict ourselves to quadratic patches, then the expressions for p_{ijk}, p_{ijkl}, etc, will include the higher order terms of the surface (d_{ijk}, d_{ijkl}, etc). In a paper in press, we have extended the results presented here to derive a unique solution in closed form for the general case of smooth surfaces that are represented by an n^{th} order Taylor series expansion (with no restriction on n).

4.1. Constraint Set

For the brightness change constraint equation to be satisfied at every point in A_z, all of the coefficients of its Taylor series expansion must vanish. That is,

$$T_{0} + m_j E_j = 0,$$

$$T_i + m_{ij} E_j + m_j E_{ji} = \mathbf{0},$$

$$T_{ij} + m_{ijk} E_k + (m_{ik} E_{kj} + m_{jk} E_{ki}) + m_k E_{kij} = \mathbf{0},$$

$$T_{ijk} + \widetilde{m}_{ij} E_k + (m_{ljk} E_{li} + m_{lik} E_{lj} + m_{lij} E_{lk}) + (m_{lk} E_{lij} + m_{lj} E_{lik} + m_{li} E_{ljk}) + m_l E_{lijk} = \mathbf{0},$$

$$
\begin{aligned}
T_{ijkl} &+ (\widetilde{m}_{ij}E_{kl} + \widetilde{m}_{ik}E_{jl} + \widetilde{m}_{il}E_{jk} + \widetilde{m}_{jk}E_{il} + \widetilde{m}_{jl}E_{ik} + \widetilde{m}_{kl}E_{ij}) + \\
&(m_{mij}E_{mkl} + m_{mik}E_{mjl} + m_{mil}E_{mjk} + m_{mjk}E_{mil} + m_{mjl}E_{mik} + m_{mkl}E_{mij} + \\
&(m_{mi}E_{mjkl} + m_{mj}E_{mikl} + m_{mk}E_{mijl} + m_{ml}E_{mijk}) + m_m E_{mijkl} = \mathbf{0}.
\end{aligned}
$$

These are 15 linear constraint equations (note that the last 3 tensor equation only give us 3, 4, and 5 constraints because of the symmetry in the indeces) in terms of the 15 essential parameters (two m_i's, three $\widetilde{m}_{ij}$'s, four m_{ij}'s, and six m_{ijk}'s). We will refer to them as a *constraint set*. Each region of analyticity will give rise to a constraint set (as opposed to a brightness constraint equation for each point in the image). Our strategy is to, first, determine the 15 essential parameters from the image derivatives using a constraint set. If several such regions exist within the quadratic patch, then a least-squares formulation is used so that we can exploit the information from more areas of the image. We then compute the motion and surface parameters from the essential parameters.

We remind the reader that there are only 11 surface and motion parameters to recover ($\{t_i, t_3\}$, $\{w_i, w_3\}$, and $\{d_i, d_{ij}\}$), however, we are measuring 15 essential parameters. We will show how the extra information will allow us to resolve the resulting ambiguities that arise in some special cases.

5. Recovering the Extended Essential Parameters

Let p be a point in A_z, and let:

$$
E_{x_i} = E_i + E_{ij}x_j + \frac{1}{2}E_{ijk}x_jx_k + \frac{1}{6}E_{ijkl}x_jx_kx_l + \frac{1}{24}E_{ijklm}x_ix_jx_kx_lx_m + \ldots.
$$

$$
E_t = T_0 + T_ix_i + \frac{1}{2}T_{ij}x_ix_j + \frac{1}{6}T_{ijk}x_ix_jx_k + \frac{1}{24}T_{ijkl}x_ix_jx_kx_l + \ldots.
$$

represent the Taylor series expansion of the intensity derivatives in A_p, a region of analyticity in A_z around the point p. Theoretically, the coefficients of these polynomials, up to fifth order terms, can be determined using the intensity data in A_p through a least-square-errors fit if the patch is strongly textured. We will show, however, that this is not necessary for determining a unique solution for the essential parameters.

Assuming these coefficients can be computed for at least one region of analyticity, A_p, we can solve for the 15 essential parameters, by elimination, from the 15 linear equations in the constraint set for region A_p:

$$
T_0 + m_jE_j = 0,
$$

$$
T_i + m_{ij}E_j + m_jE_{ji} = \mathbf{0},
$$

$$
T_{ij} + m_{ijk}E_k + (m_{ik}E_{kj} + m_{jk}E_{ki}) + m_kE_{kij} = \mathbf{0},
$$

$$
T_{ijk} + \widetilde{m}_{ij}E_k + (m_{ljk}E_{li} + m_{lik}E_{lj} + m_{lij}E_{lk}) + (m_{lk}E_{lij} + m_{lj}E_{lik} + m_{li}E_{ljk}) + m_lE_{lijk} = \mathbf{0},
$$

$$T_{ijkl}+(\tilde{m}_{ij}E_{kl}+\tilde{m}_{ik}E_{jl}+\tilde{m}_{il}E_{jk}+\tilde{m}_{jk}E_{il}+\tilde{m}_{jl}E_{ik}+\tilde{m}_{kl}E_{ij})$$
$$+(m_{mij}E_{mkl}+m_{mik}E_{mjl}+m_{mil}E_{mjk}+m_{mjk}E_{mil}+m_{mjl}E_{mik}+m_{mkl}E_{mij})$$
$$+(m_{mi}E_{mjkl}+m_{mj}E_{mikl}+m_{mk}E_{mijl}+m_{ml}E_{mijk})+m_mE_{mijkl}=\mathbf{0}.$$

When several regions of analyticity exist on the surface patch, then a least-squares formulation should be implemented to exploit the information from every region.

As we said above, it is not necessary to compute up to the fifth order terms of the intensity function. We can get an approximate solution from:

$$T_{\shortparallel}+m_jE_j=\mathbf{0},$$

$$T_i+m_{ij}E_j+m_jE_{ji}=\mathbf{0},$$

$$T_{ij}+m_{ijk}E_k+(m_{ik}E_{kj}+m_{jk}E_{ki})+m_kE_{kij}=\mathbf{0},$$

$$T_{ijk}+\tilde{m}_{ij}E_k+(m_{ljk}E_{li}+m_{lik}E_{lj}+m_{lij}E_{lk})+(m_{lk}E_{lij}+m_{lj}E_{lik}+m_{li}E_{ljk})=\mathbf{0},$$

$$(\tilde{m}_{ij}E_{kl}+\tilde{m}_{ik}E_{jl}+\tilde{m}_{il}E_{jk}+\tilde{m}_{jk}E_{il}+\tilde{m}_{jl}E_{ik}+\tilde{m}_{kl}E_{ij})+$$
$$(m_{mij}E_{mkl}+m_{mik}E_{mjl}+m_{mil}E_{mjk}+m_{mjk}E_{mil}+m_{mjl}E_{mik}+m_{mkl}E_{mij})=\mathbf{0},$$

that is, using only up to the third order terms of the intensity function. Similarly we may drop the third order terms and use the resulting equations to compute the essential parameters. In essence, we are compromising the accuracy of the solution of the essential parameters for two objectives. First, it is more practical to fit quadratic or cubic polynomials to intensity function in a small neighborhood, and secondly, less computation is required to do so.

We will show that in all but two cases, m_i, m_{ij}, and m_{ijk} are sufficient to derive uniquly a closed form solution for the motion and surface parameters. For these two cases, we have to resort to the constraints given by $\tilde{m}_{ij}$ to resolve the two-fold ambiguities in the solution. In other words, we can use the equations for m_i, m_{ij}, and m_{ijk} to compute $\{w_i, w_3\}$, $\{t_i, t_3\}$, and $\{d_i, d_{ij}\}$ uniquely except for two special cases. when these arise, then we use the equations for $\tilde{m}_{ij}$ to resolve the resulting two-fold ambiguities.

Since the two special cases may occur rarely, we can consider a different strategy that does not require the computation of $\tilde{m}_{ij}$. Since $\tilde{m}_{ij}$ only appears in the last two tensor equations, we consider only the following equations:

$$T_{\shortparallel}+m_jE_j=\mathbf{0},$$

$$T_i+m_{ij}E_j+m_jE_{ji}=\mathbf{0},$$

$$T_{ij}+m_{ijk}E_k+(m_{ik}E_{kj}+m_{jk}E_{ki})+m_kE_{kij}=\mathbf{0}.$$

We can use these equations to compute the remaining essential parameters, however, we only have 6 constraint equations, for each point p, in terms of the 12 unknown essential

parameters. So we need at least two points. Using two points, we can solve for m_i as follows:

$$m_i = -G_{pi}T_0^p,$$

where

$$G_{pi} = \frac{1}{\epsilon_{jk}E_j^1 E_k^2}\begin{pmatrix} E_2^2 & -E_2^1 \\ -E_1^2 & E_1^1 \end{pmatrix},$$

and superscript p denotes the coefficients associated with point p. Using the solution for m_i, we can now solve the second tensor equation for m_{ij}. The solution is given by:

$$m_{ij} = -G_{pj}(T_i^p + E_{ki}^p m_k).$$

Finally, using m_i and m_{ij} from above, the solution for m_{ijk} can be expressed by:

$$m_{ijk} = -G_{pk}(T_{ij}^p + E_{lj}^p m_{il} + E_{li}^p m_{jl} + m_l E_{lij}^p).$$

It is now clear why we need to consider two points from two different regions of analyticity. To be able to solve for the essential parameters, it is necessary and sufficient that:

$$\epsilon_{ij}E_i^1 E_j^2 \neq 0.$$

If the two points are chosen from the same region, the coefficients of the Taylor series expansion of the intensity function about each point will be the same, and therefore:

$$\epsilon_{ij}E_i^1 E_j^2 = 0.$$

It is appropriate to point out what information is necessary to recover the essential parameters. It is necessary to compute the first order coefficients of the image function, however, the higher order terms only provide more accurate estimates of the unknown parameters. To show this, let us assume that only the first order terms are available. Then, we can still determine the essential parameters from:

$$m_i = -G_{pi}T_0^p, \qquad m_{ij} = -G_{pj}T_i^p, \qquad m_{ijk} = 0.$$

The terms we neglected could be viewed as the correction terms that come from the higher order information. We can still estimate the motion and surface parameters without them, even though these estimates may not be very accurate. Similarly, we can get an approximate solution for m_{ijk} by neglecting the last term involving E_{lij}. In this case, we only have to compute up to the second order terms of the Taylor series expansions of the intensity function and its temporal derivative in each region of analyticity. In practice, this is desired not only from a computational consideration, but also with noisy images, one cannot determine these coefficients accurately.

Image brightness values are corrupted with sensor noise and quantization. So it is not advisable to base a method on measurements at just a few points. Instead we propose a

least-square-errors formulation. We choose m_i, m_{ij}, and m_{ijk} that minimize:

$$J = \frac{1}{2}\int_{A_z} \Big((T_0 + m_i E_i)^2 + (T_i + m_{ij} E_j + m_j E_{ji})^2 + \\ \big(T_{ij} + m_{ijk} E_k + (m_{ik}\delta_{mj} + m_{jk}\delta_{mi}) E_{km} + m_l E_{lij}\big)^2 \Big) dA,$$

over the whole region A_z. For an extremum of J, we must have:

$$\frac{\partial J}{\partial m_i} = 0, \quad \frac{\partial J}{\partial m_{ij}} = 0, \quad \text{and} \quad \frac{\partial J}{\partial m_{ijk}} = 0.$$

After differentiating the expression for J with respect to the essential parameters, and performing some algebraic manipulations, we arrive at:

$$\begin{aligned} A_{ik} m_k + A_{ikl} m_{kl} + A_{iklm} m_{klm} &= -A_i, \\ B_{ijk} m_k + B_{ijkl} m_{kl} + B_{ijklm} m_{klm} &= -B_{ij}, \\ C_{ijkl} m_l + C_{ijklm} m_{lm} + C_{kl} m_{ijl} &= -C_{ijk}. \end{aligned}$$

where

$$A_i = \int_{A_z} (T_0 E_i + T_l E_{il} + 2T_{ij} E_j + T_{lj} E_{ilj}) dA,$$

$$A_{ik} = \int_{A_z} (E_i E_k + E_{ij} Ekj + \delta_{ki} E_{lpq} E_{lpq}) dA,$$

$$A_{ikl} = \int_{A_z} (E_l E_{ik} + 2E_{kj} E_{ilj}) dA, \quad A_{iklm} = \int_{A_z} (E_k E_{ilm}) dA,$$

$$B_{ij} = \int_{A_z} (T_i E_j + 2T_{ik} E_{jk}) dA, \quad B_{ijk} = \int_{A_z} (E_{ki} E_j + 2E_{jl} E_{kil}) dA,$$

$$B_{ijkl} = \int_{A_z} (\delta_{ik} E_l E_j + 2\delta_{ik} E_{lm} E_{jm} + 2E_{jk} E_{li}) dA,$$

$$B_{ijklm} = \int_{A_z} (\delta_{ki} E_{jl} + \delta_{li} E_{jk}) E_m dA,$$

$$C_{ijk} = \int_{A_z} (T_{ij} E_k) dA, \quad C_{ijkl} = \int_{A_z} (E_k E_{lij}) dA,$$

$$C_{ijklm} = \int_{A_z} (\delta_{il} E_{mj} E_k + \delta_{jl} E_{mi} E_k) dA, \quad C_{kl} = \int_{A_z} (E_k E_l) dA,$$

These twelve linear equations can easily be solved for the optimum m_i, m_{ij}, and m_{ijk} by elimination. The derivation of the results is a tedious practice of elementary algebra and is omitted here (note that the last tensor equation only gives 6 independent scalar equations due to the symmetry in i and j).

So in summary, we have shown how to compute the essential parameters from just one point (using the fifth order terms of intensity function in the region of analyticity containing the point), or two points (using third order terms of intensity function in the region of analyticity containing the point). We also showed that approximate results may be obtained if one uses just the lower order terms. When more points (or correspondingly more regions of analyticity) are available, a least-squares formulation can be employed that exploits the information from larger portions of the image plane.

In most cases, it will be enough to compute up to second order terms of the Taylor series expansion of the image function, however, as explained earlier, we have to resort to the information provided by the higher order terms to resolve the resulting two-fold ambiguities in the solution for two special cases.

6. Recovering the Motion and Surface Parameters

The essential parameters were defined in terms of some nonlinear functions of the motion and surface parameters, $\{t_i, t_3\}$, $\{w_i, w_3\}$, and $\{d_i, d_{ij}\}$:

$$m_i = \epsilon_{ji} w_j - t_i,$$

$$m_{ij} = \delta_{ij} t_3 - \epsilon_{ij} w_3 - d_i t_j, \quad \widetilde{m}_{ij} = t_3 d_{ij}.$$

$$m_{ijk} = (\epsilon_{lj} w_l + d_j t_3)\delta_{ki} + (\epsilon_{li} w_l + d_i t_3)\delta_{kj} - t_k d_{ij}.$$

We will show that the equations for m_i, m_{ij}, m_{ijk} can be used to recover the motion and surface parameters uniquely in all but two cases. When there is no component of motion parallel to the image plane d_{ij} (these are related to the curvature of the surface around the fixation point) cannot be recovered, and therefore, we have to contend with a two-fold ambiguity similar to the one associated with planar surfaces in motion, and when there is no motion in the direction of zero surface gradient, there is yet another two-fold ambiguity. It is in these cases that the equations for $\widetilde{m}_{ij}$ come to rescue. As a result, we are able to recover the motion and surface parameters uniquely in every case. So we will not use the equations for $\widetilde{m}_{ij}$ until the very end. We choose this strategy since to compute $\widetilde{m}_{ij}$, as we showed earlier, we need good estimates of the higher order terms in the Taylor series expansion of the image intensity function. In practice, these terms cannot be computed reliably from noisy images.

It is possible to solve the above non-linear equations for the motion and surface parameters in several ways. Curiously, they all reduce to solving a cubic equation in terms of t_1/t_2 (or t_2/t_1) that was first derived by Longuet-Higgins and Prazdny [] in their work using optical flow (however, they never tried to solve it analytically, and so concluded that, in general, three solutions are possible). Once the cubic equation is solved, the unknown parameters can be determined easily.

It was brought to our attention that Waxman et al. [25] had derived the same results from optical flow data using the formulation originally proposed in Waxman and Ullman

[23], and by exploiting 12 *observables* that are actually linear combinations of our 12 essential parameters m_i, m_{ij}, and m_{ijk}), that are computed from the optical flow and its *first* and *second* derivatives. Unfortunately, their formulation, using the 12 observables, does not allow them to resolve the resulting two-fold ambiguities in the two special cases.

Here, we present a procedure that uses the image coordinate transformation that Longuet-Higgins and Prazdny suggested. Before we proceed, we will redefine m_{ijk} by subtracting twice the expression for m_i from the old definition of m_{ijk}. The new equations will be given by:

$$m_i = \epsilon_{ji} w_j - t_i,$$

$$m_{ij} = \delta_{ij} t_3 - \epsilon_{ij} w_3 - d_i t_j, \quad \widetilde{m}_{ij} = t_3 d_{ij}.$$

$$m_{ijk} = (t_j + d_j t_3)\delta_{ki} + (t_i + d_i t_3)\delta_{kj} - t_k d_{ij}.$$

An image coordinate transformation can be written as:

$$x'_i = \Omega_{ij} x_j = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x_j.$$

We will explain how θ is chosen later. Let m'_i, m'_{ij}, and m'_{ijk} denote the extended essential parameters in the new coordinate system. Then:

$$m'_i = \Omega_{il} m_l = \Omega_{il}(\epsilon_{jl} w_j - t_l),$$

$$m'_{ij} = \Omega_{il}\Omega_{jm} m_{lm} = \Omega_{il}\Omega_{jm}(\delta_{lm} t_3 - \epsilon_{lm} w_3 - d_l t_m),$$

$$m'_{ijk} = \Omega_{il}\Omega_{jm}\Omega_{kn} m_{lmn} = \Omega_{il}\Omega_{jm}\Omega_{kn}\Big((t_m + d_m t_3)\delta_{nl} + (t_l + d_l t_3)\delta_{nm} - t_n d_{lm}\Big).$$

These can be written in the form:

$$m'_i = \epsilon_{ji} w'_j - t'_i,$$

$$m'_{ij} = \delta_{ij} t'_3 - \epsilon_{ij} w'_3 - d'_i t'_j,$$

$$m'_{ijk} = \delta_{ki}(t'_j + d'_j t'_3) + \delta_{kj}(t'_i + d'_i t'_3) - t'_k d'_{ij},$$

where,

$$w'_i = \Omega_{il} w_l, \quad t'_i = \Omega_{il} t_l, \qquad d'_i = \Omega_{il} d_l, \quad d'_{ij} = \Omega_{il}\Omega_{jm} d_{lm}.$$

Since the rotation to the new coordinate system is about the z-axis, it is clear that $t_3 = t'_3$, and $w_3 = w'_3$.

Let us write the equations for the transformed extended essential parameters in the component form:

$$\begin{array}{ll} m'_1 = -w'_2 - t'_1 & m'_2 = w'_1 - t'_2 \\ m'_{11} = t'_3 - d'_1 t'_1 & m'_{22} = t'_3 - d'_2 t'_2 \\ m'_{12} = -w'_3 - d'_1 t'_2 & m'_{21} = w'_3 - d'_2 t'_1 \\ m'_{111} = 2(t'_1 + d'_1 t'_3) - t'_1 d'_{11} & m'_{222} = 2(t'_2 + d'_2 t'_3) - t'_2 d'_{22} \end{array}$$

$$\begin{array}{ll} m'_{112} = -t'_2 d'_{11} & m'_{221} = -t'_1 d'_{22} \\ m'_{121} = t'_2 + d'_2 t'_3 - t'_1 d'_{12} & m'_{122} = t'_1 + d'_1 t'_3 - t'_2 d'_{12} \end{array}$$

The new coordinate system is chosen such that the x'_2-axis (or alternatively, x'_1-axis, due to symmetry in the above equations) is aligned with the component of the translation in the image plane so that $t'_1 = 0$ (if we align the x'_1-axis with t_i, then $t'_2 = 0$). Note that θ is not known, since t_i is unknown. We will show how to compute θ analytically.

Setting $t'_1 = 0$, we arrive at:

$$\begin{array}{ll} m'_1 = -w'_2 & m'_2 = w'_1 - t'_2 \\ m'_{11} = t'_3 & m'_{22} = t'_3 - d'_2 t'_2 \\ m'_{12} = -w'_3 - d'_1 t'_2 & m'_{21} = w'_3 \\ m'_{111} = 2d'_1 t'_3 & m'_{222} = 2(t'_2 + d'_2 t'_3) - t'_2 d'_{22} \\ m'_{112} = -t'_2 d'_{11} & m'_{221} = 0 \\ m'_{121} = t'_2 + d'_2 t'_3 & m'_{122} = d'_1 t'_3 - t'_2 d'_{12}. \end{array}$$

We use the constraint $m'_{221} = 0$ to solve for θ. Writing this in terms of the original parameters, we obtain:

$$m'_{221} = \Omega_{2l}\Omega_{2m}\Omega_{1n} m_{lmn} = 0.$$

After substituting for Ω_{ij}'s, collecting terms and simplifying, we arrive at:

$$-m_{112}\tan^3\theta + (m_{111} - 2m_{122})\tan^2\theta - (m_{222} - 2m_{121})\tan\theta + m_{221} = 0,$$

or in terms of the motion and surface parameters,

$$t_2 d_{11} \tan^3\theta + (2t_2 d_{12} - t_1 d_{11})\tan^2\theta + (t_2 d_{22} - 2t_1 d_{12})\tan\theta - t_1 d_{22} = 0,$$

which can be factored into the following form:

$$(t_2 \tan\theta - t_1)(d_{11}\tan^2\theta + 2d_{12}\tan\theta + d_{22}) = 0.$$

The solutions of the above equation are given by:

$$\tan\theta = \frac{t_1}{t_2}, \quad \frac{-d_{12} \pm \sqrt{d_{12}^2 - d_{11}d_{22}}}{d_{11}}.$$

The cubic equation is trivially satisfied for two special cases: $d_{ij} = \mathbf{0}$, that is, the surface is planar, or $t_i = \mathbf{0}$. If the coefficients of the cubic equation do not vanish, we are guaranteed that:

$$t_i \neq \mathbf{0}, \quad \text{and} \quad d_{ij} \neq \mathbf{0}.$$

For now, we assume these conditions are satisfied.

Note that only the first solution, $\tan\theta = t_1/t_2$, corresponds to the correct transformation. Let us look at the other two solutions, i.e., the solutions of:

$$d_{11}\tan^2\theta + 2d_{12}\tan\theta + d_{22} = 0.$$

Remember that:

$$d_{ij} = -(\frac{\partial^2 Z}{\partial X_i \partial X_j})_o,$$

and therefore,

$$(d_{11} \tan\theta^2 + 2d_{12}\tan\theta + d_{22}) = -v_i H_{ij} v_j,$$

where

$$v_i = (\tan\theta \quad 1),$$

$$H_{ij} = \begin{pmatrix} \frac{\partial^2 Z}{\partial X_1^2} & \frac{\partial^2 Z}{\partial X_1 \partial X_2} \\ \frac{\partial^2 Z}{\partial X_1 \partial X_2} & \frac{\partial^2 Z}{\partial X_2^2} \end{pmatrix}.$$

As shown in appendix 1, if the surface has a positive gaussian curvature at O, that is O is an *elliptical paraboloid* point, then H_{ij} is either positive or negative definite, and so:

$$d_{11}\tan^2\theta + 2d_{12}\tan\theta + d_{22} = -v_i H_{ij} v_j \neq 0.$$

Therefore, the solution for the transformation angle θ is unique . If the gaussian curvature at O is zero but $d_{ij} \neq \mathbf{0}$, or if O is a *parabolic cylinder* point, the above quadratic equation has two identical solutions given by:

$$\tan\theta = -\frac{d_{12}}{d_{11}} = -\frac{d_{22}}{d_{12}}.$$

Finally if the surface has a negative gaussian curvature at O (*hyperbolic paraboloid point*), then the quadratic equation has two distinct real solutions that are:

$$\tan\theta = \frac{-d_{12} \pm \sqrt{d_{12}^2 - d_{11}d_{22}}}{d_{11}}.$$

So the solution for the transformation angle is not unique when point O is either a cylindrical paraboloid or a hyperbolic paraboloid point. We will show that, in most cases, it is possible to single out the correct solution, $\theta = \tan^{-1} t_1/t_2$, using the extra constraint equations.

Once we solve for the transformation angle(s), we can determine the transformed parameters, m'_i, m'_{ij}, and m'_{ijk}, and determine the motion and surface parameters from:

$$\begin{aligned} m'_1 &= -w'_2 & m'_2 &= w'_1 - t'_2 \\ m'_{11} &= t'_3 & m'_{22} &= t'_3 - d'_2 t'_2 \\ m'_{12} &= -w'_3 - d'_1 t'_2 & m'_{21} &= w'_3 \\ m'_{111} &= 2d'_1 t'_3 & m'_{222} &= 2(t'_2 + d'_2 t'_3) - t'_2 d'_{22} \\ m'_{112} &= -t'_2 d'_{11} & m'_{221} &= 0 \\ m'_{121} &= t'_2 + d'_2 t'_3 & m'_{122} &= d'_1 t'_3 - t'_2 d'_{12}. \end{aligned}$$

If $t_3 = t'_3 = m'_{11} = 0$, the solution is given by:

$$\begin{array}{lll} w'_1 = m'_2 + m'_{121} & w'_2 = -m'_1 & w'_3 = m'_{21} \\ t'_1 = 0 & t'_2 = m'_{121} & t'_3 = m'_{11} = 0 \\ d'_0 = 1 & d'_1 = -\dfrac{m'_{21} + m'_{12}}{m'_{121}} & d'_2 = -\dfrac{m'_{22}}{m'_{121}} \\ d'_{11} = -\dfrac{m'_{112}}{m'_{121}} & d'_{12} = -\dfrac{m'_{122}}{m'_{121}} & d'_{22} = -\dfrac{m'_{222} - 2m'_{121}}{m'_{121}} \end{array}$$

If $t_3 = t'_3 = m'_{11} \neq 0$, we determine w'_2, w'_3, and t'_3 as before:

$$w'_2 = -m'_1, \qquad , w'_3 = m'_{21}, \qquad t'_3 = m'_{11}.$$

Substituting for these into the remaining equations, we obtain:

$$\begin{aligned} m'_2 &= w'_1 - t'_2, \\ m'_{22} &= m'_{11} - d'_2 t'_2, \\ m'_{12} &= -m'_{21} - d'_1 t'_2, \\ m'_{111} &= 2m'_{11} d'_1, \\ m'_{222} &= 2(t'_2 + m'_{11} d'_2) - t'_2 d'_{22}, \\ m'_{112} &= -t'_2 d'_{11}, \\ m'_{121} &= t'_2 + m'_{11} d'_2, \\ m'_{122} &= m'_{11} d'_1 - t'_2 d'_{12}. \end{aligned}$$

We can solve for d'_1 from:

$$d'_1 = \frac{m'_{111}}{2m'_{11}}.$$

Now expressing d'_2 in terms of t'_2 we get:

$$d'_2 = \frac{m'_{121} - t'_2}{m'_{11}},$$

and substituting into the equation for m'_{22}, we arrive at:

$$m'_{22} = m'_{11} - \frac{t'_2(m'_{121} - t'_2)}{m'_{11}}.$$

This simplifies into:

$$t'^2_2 - m'_{121} t'_2 + m'_{11}(m'_{11} - m'_{22}) = 0.$$

The two possible solution for t'_2 are given by:

$$t'_2 = \frac{1}{2}\left(m'_{121} \pm \sqrt{m'^2_{121} - 4m'_{11}(m'_{11} - m'_{22})}\right),$$

or in terms of the motion and surface parameters:

$$t_2' = t_2', \qquad \text{and} \qquad d_2't_3'.$$

We now proceed to determine the solution for the remaining parameters in terms of the solutions given for t_2':

$$w_1' = m_2' + t_2',$$

$$d_{11}' = -\frac{m_{112}'}{t_2'}, \quad d_{12}' = -\frac{2m_{122}' - m_{111}'}{2t_2'}, \quad d_{22}' = -\frac{m_{222}' - 2m_{121}'}{t_2'}.$$

Now consider the extra equation we have not used yet, i.e.,

$$m_{12}' + m_{21}' = -d_1't_2'.$$

In most cases, of the two possible solutions of t_2' given earlier, only the corect one satisfies this constraint equation. So the correct solution for t_2' (that will be unique) is chosen and used to determine the other parameters. Similarly, in most cases, this equality will not hold for the wrong transformation angle θ. So the above constraint can be used to determine the correct solution for t_2' (from the dual set) and the proper transformation angle (if the cubic equation we derived earlier has multiple solutions).

In the special case when $d_1' = 0$, we have:

$$m_{12}' + m_{21}' = 0.$$

So the extra equation becomes independent of t_2', and cannot be used to determine the correct solution of t_2'. However, the constraint equations:

$$\tilde{m}_{ij}' = t_3'd_{ij}',$$

come to rescue here. Of the two solutions for t_2 that result in:

$$d_{11}' = -\frac{m_{112}'}{t_2'}, \quad d_{12}' = -\frac{2m_{122}' - m_{111}'}{2t_2'}, \quad d_{22}' = -\frac{m_{222}' - 2m_{121}'}{t_2'},$$

only one is consistant with the solution obtained from:

$$d_{11}' = -\frac{\tilde{m}_{11}'}{m_{11}'}, \quad d_{12}' = -\frac{\tilde{m}_{12}'}{m_{11}'}, \quad d_{22}' = -\frac{\tilde{m}_{22}'}{m_{11}'}.$$

We again emphasize that the constraints from $\tilde{m}_{ij}$ are only used for removing the ambiguities, and not to compute the unknown parameters for the reasons given earlier.

If $d_i' = 0$, the ambiguity is resolved without the need for the constraints from $\tilde{m}_{ij}$. This is because the dual solution for t_2' $(d_2't_3')$ will be zero, and we can discard it since we assumed that $t_2' \neq 0$ ($t_i' \neq 0$ and $t_1' = 0$ imply that $t_2' \neq 0$) .

Now we consider the case when the coefficients of the cubic equation vanish. Then, either $t_i = \mathbf{0}$, or $d_{ij} = \mathbf{0}$. Again, we use the constraints:

$$\tilde{m}'_{ij} = t'_3 d'_{ij},$$

to distinguish between the two cases, and determine the solution.

If $\tilde{m}'_{ij} = \mathbf{0}$, then $d'_{ij} = \mathbf{0}$ since we assumed that $t'_3 \neq 0$ (the case $t'_3 = 0$ was treated earlier). The remaining parameters can be determined using the method we proposed in [16], however, the two-fold ambiguity of planar surfaces is what we have to contend with.

If $\tilde{m}'_{ij} \neq \mathbf{0}$, then we know that $d'_{ij} \neq \mathbf{0}$, and therefore, $t'_i = \mathbf{0}$. In this case, the solution is given by:

$$\begin{array}{lll}
w'_1 = m'_2 & w'_2 = -m'_1 & w'_3 = m'_{21} = -m'_{12} \\
t'_1 = 0 & t'_2 = 0 & t'_3 = m'_{11} = m'_{22} \\
d'_0 = 1 & d'_1 = -\dfrac{m'_{111}}{2m'_{11}} = -\dfrac{m'_{111}}{2m'_{22}} & d'_2 = -\dfrac{m'_{222}}{2m'_{11}} = -\dfrac{m'_{222}}{2m'_{22}} \\
d'_{11} = \dfrac{\tilde{m}'_{11}}{m'_{11}} = \dfrac{\tilde{m}'_{11}}{m'_{22}} & d'_{12} = \dfrac{\tilde{m}'_{12}}{m'_{11}} = \dfrac{\tilde{m}'_{12}}{m'_{22}} & d'_{22} = \dfrac{\tilde{m}'_{22}}{m'_{11}} = \dfrac{\tilde{m}'_{22}}{m'_{22}}
\end{array}$$

This is the only case we actually need $\tilde{m}_{ij}$ to estimate the surface parameters d_{ij}.

So in summary, we proceed as follows:

(1) We determine the solution(s) for the transformation angle from:

$$-m_{112}\tan^3\theta + (m_{111} - 2m_{122})\tan^2\theta - (2m_{121} - m_{222})\tan\theta + m_{221} = 0.$$

If all of the coefficients of the above equation vanish, then either $d_{ij} = d'_{ij} = \mathbf{0}$, or $t_i = t'_i = \mathbf{0}$. We use the constraint equations:

$$\tilde{m}'_{ij} = t'_3 d'_{ij},$$

to distinguish between the two cases. If $\tilde{m}'_{ij} = \mathbf{0}$, we conclude that $d'_{ij} = \mathbf{0}$ and therefore, the surface is planar. We can use our method for planar surfaces (see [16]) to compute the unknown parameters. we have to contend with the well-known two-fold ambiguity of planar surfaces. If $\tilde{m}'_{ij} \neq 0$, then the solution is given by:

$$\begin{array}{lll}
w'_1 = m'_2 & w'_2 = -m'_1 & w'_3 = m'_{21} = -m'_{12} \\
t'_1 = 0 & t'_2 = 0 & t'_3 = m'_{11} = m'_{22} \\
d'_0 = 1 & d'_1 = -\dfrac{m'_{111}}{2m'_{11}} = -\dfrac{m'_{111}}{2m'_{22}} & d'_2 = -\dfrac{m'_{222}}{2m'_{11}} = -\dfrac{m'_{222}}{2m'_{22}} \\
d'_{11} = \dfrac{\tilde{m}'_{11}}{m'_{11}} = \dfrac{\tilde{m}'_{11}}{m'_{22}} & d'_{12} = \dfrac{\tilde{m}'_{12}}{m'_{11}} = \dfrac{\tilde{m}'_{12}}{m'_{22}} & d'_{22} = \dfrac{\tilde{m}'_{22}}{m'_{11}} = \dfrac{\tilde{m}'_{22}}{m'_{22}}
\end{array}$$

When the coefficients of the cubic do not vanish, we conclude that:

$$t_i = t'_i \neq \mathbf{0}, \qquad \text{and} \qquad d_{ij} = d'_{ij} \neq \mathbf{0}.$$

For each solution of θ, we compute the extended essential parameters in the rotated coordinate frame.

(2) We determine the solution for t'_3 from $t'_3 = m'_{11}$. If $t'_3 = 0$, then the solution for the motion and surface parameters is unique and is given by:

$$\begin{array}{lll} w'_1 = m'_2 + m'_{121} & w'_2 = -m'_1 & w'_3 = m'_{21} \\ t'_1 = 0 & t'_2 = m'_{121} & t'_3 = m'_{11} = 0 \\ d'_0 = 1 & d'_1 = -\dfrac{m'_{21} + m'_{12}}{m'_{121}} & d'_2 = -\dfrac{m'_{22}}{m'_{121}} \\ d'_{11} = -\dfrac{m'_{112}}{m'_{121}} & d'_{12} = -\dfrac{m'_{122}}{m'_{121}} & d'_{22} = -\dfrac{m'_{222} - 2m'_{121}}{m'_{121}} \end{array}$$

If $t'_3 \neq 0$, we first find the two solutions for t'_2 from:

$$t'_2 = \frac{1}{2}\left(m'_{121} \pm \sqrt{m'_{121} - 4m'_{11}(m'_{11} - m'_{22})}\right).$$

If one of the solutions is zero, we discard it (resolvable ambiguous case), and proceed with the correct solution. If both solutions are nonzero, we pick the correct one that satisfies:

$$m'_{12} + m'_{21} = -d'_1 t'_2.$$

If this constraint is trivially satisfied, that is:

$$m'_{12} + m'_{21} = 0,$$

we conclude that $d'_1 = 0$. Of the two solutions given by:

$$\begin{array}{lll} w'_1 = m'_2 + t'_2 & w'_2 = -m'_1 & w'_3 = m'_{21} \\ t'_1 = 0 & t'_2 = t'_2, \text{ and } d'_2 t'_3 & t'_3 = m'_{11} \\ d'_0 = 1 & d'_1 = \dfrac{m'_{111}}{2m'_{11}} & d'_2 = \dfrac{m'_{121} - t'_2}{m'_{11}} \\ d'_{11} = -\dfrac{m'_{112}}{t'_2} & d'_{12} = -\dfrac{2m'_{122} - m'_{111}}{2t'_2} & d'_{22} = -\dfrac{m'_{222} - 2m'_{121}}{t'_2} \end{array}$$

we select the one that is consistent with the solution obtained from:

$$\tilde{m}'_{ij} = t'_3 d'_{ij},$$

that is,

$$d'_{11} = \frac{\tilde{m}'_{11}}{m'_{11}}, \quad d'_{12} = \frac{\tilde{m}'_{12}}{m'_{11}}, \quad d'_{22} = \frac{\tilde{m}'_{22}}{m'_{11}}.$$

(3) In most cases, the multiple ambiguity in the transformation angle, if it exists, can be resolved by the extra constraint equation:

$$m'_{12} + m'_{21} = 0.$$

In general, this constraint will be satisfied only for the correct rotation. We restate that these ambiguities occur only when the gaussian curvature at O is non-positive (the fixation point is either a parabolic cylinder or a hyperbolic paraboloid point). For surfaces with positive gaussian curvature at O, the solution for the transformation angle is always unique. If we have not resolved the angle ambiguity yet, we, once again, turn to:

$$\tilde{m}'_{ij} = t'_3 d'_{ij},$$

since these constraints should only be satisfied for the correct transformation. It is very unlikely that the incorrect transformation satisfisfies all of the above constraints.

(4) We then compute the motion and surface parameters in the original coordinate frame using the transformation equations given by:

$$w_l = \Omega_{il} w'_i, \quad t_l = \Omega_{il} t'_i, \quad d_l = \Omega_{il} d'_i, \quad d_{lm} = \Omega_{il} \Omega_{jm} d'_{ij}.$$

7. Summary

In this paper, we have investigated the problem of recovering the motion of an observer relative to a stationary environment directly from the time-varying imagery. We assume that, in a small neighborhood, the surface of the scene can be approximated by a quadratic patch. We first compute 15 intermediate parameters termed *essential parameters* from a set of linear constraint equations in terms of the image intensity derivatives. We then solve for the observer motion parameters and the local structure of the scene (slope and curvature parameters) in closed form.

We have shown that the solution is unique in every case. The previous results of Longuet-Higgins and Prazdny [10], or that of Waxman et al. [23,25] fall short of showing uniqueness for every case. Furthermore, they use the optical flow data. So far, no method for computing a correct optical flow is proposed in the literature than can be reliably used for the optical flow based methods.

In a paper in press [17], we consider smooth textured surfaces with continuous n^{th} order derivatives (no restriction on n). We show that the problem of recovering the motion and the coefficients of the surface can be decoupled into two problems: (1) determining the motion and the first and second order parameters of the surface through the procedure presented in this paper, and (2) using the solution from the first problem to solve for the higher order coefficients from a set of linear constraints. This suggests that including the higher order terms of the surface is not necessary for recovering the motion parameters.

The uniqueness results we derive have important implications on the interpretation of the motion of objects with curved surfaces.

8. Acknowledgement

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Appendix 1:

Earlier, we presented a procedure of recovering the motion of an observer relative to a locally quadratic patch. It is based on solving a set of nonlinear equations in a transformed image coordinate frame. To compute the transformation angle, we need to solve the following cubic equation:

$$(t_2 \tan\theta - t_1)(d_{11}\tan^2\theta + 2d_{12}\tan\theta + d_{22}) = 0.$$

Only one of the solutions of this equation, $\tan\theta = t_1/t_2$, corresponds to the proper rotation. The other two solutions:

$$\tan\theta = \frac{-d_{12} \pm \sqrt{d_{12}^2 - d_{11}d_{22}}}{d_{11}},$$

if they exist, are directly related to the local properties of the surface, namely the gaussian curvature of the surface, at 0 (fixation point). Note that we are only interested in the real solutions of the above quadratic solution, and these may not exist. Here, we will investigate these in more details.

As before, let $v_i = (\tan\theta \quad 1)$, and let:

$$H_{ij} = -d_{ij} = \begin{pmatrix} \frac{\partial^2 Z}{\partial X_1^2} & \frac{\partial^2 Z}{\partial X_1 \partial X_2} \\ \frac{\partial^2 Z}{\partial X_1 \partial X_2} & \frac{\partial^2 Z}{\partial X_2^2} \end{pmatrix}$$

represent the Hessian of the surface at O. The sign of the guassian curvature of the surface at this point is that of $|H_{ij}|$, where:

$$|H_{ij}| = d_{11}d_{22} - d_{12}^2.$$

If $|H_{ij}| > 0$, that is, the surface has a positive guassian curvature at O (see Figure 2a), then it is clear that the two solutions of θ:

$$\tan\theta = \frac{-d_{12} \pm \sqrt{d_{12}^2 - d_{11}d_{22}}}{d_{11}}$$

are complex. The fixation point O, in this case, is referred to as a *eliptic paraboloid point* in the literature.

Now consider the case when H_{ij} is singular, that is $|H_{ij}| = 0$, but $d_{ij} \neq \mathbf{0}$. In this case, the guassian curvature, at O is zero (see Figure 2b). In this case (O is a *parabolic cylinder* point), the two real solutions of θ become identical, given by:

$$\tan\theta = -\frac{d_{12}}{d_{11}} = -\frac{d_{22}}{d_{12}}.$$

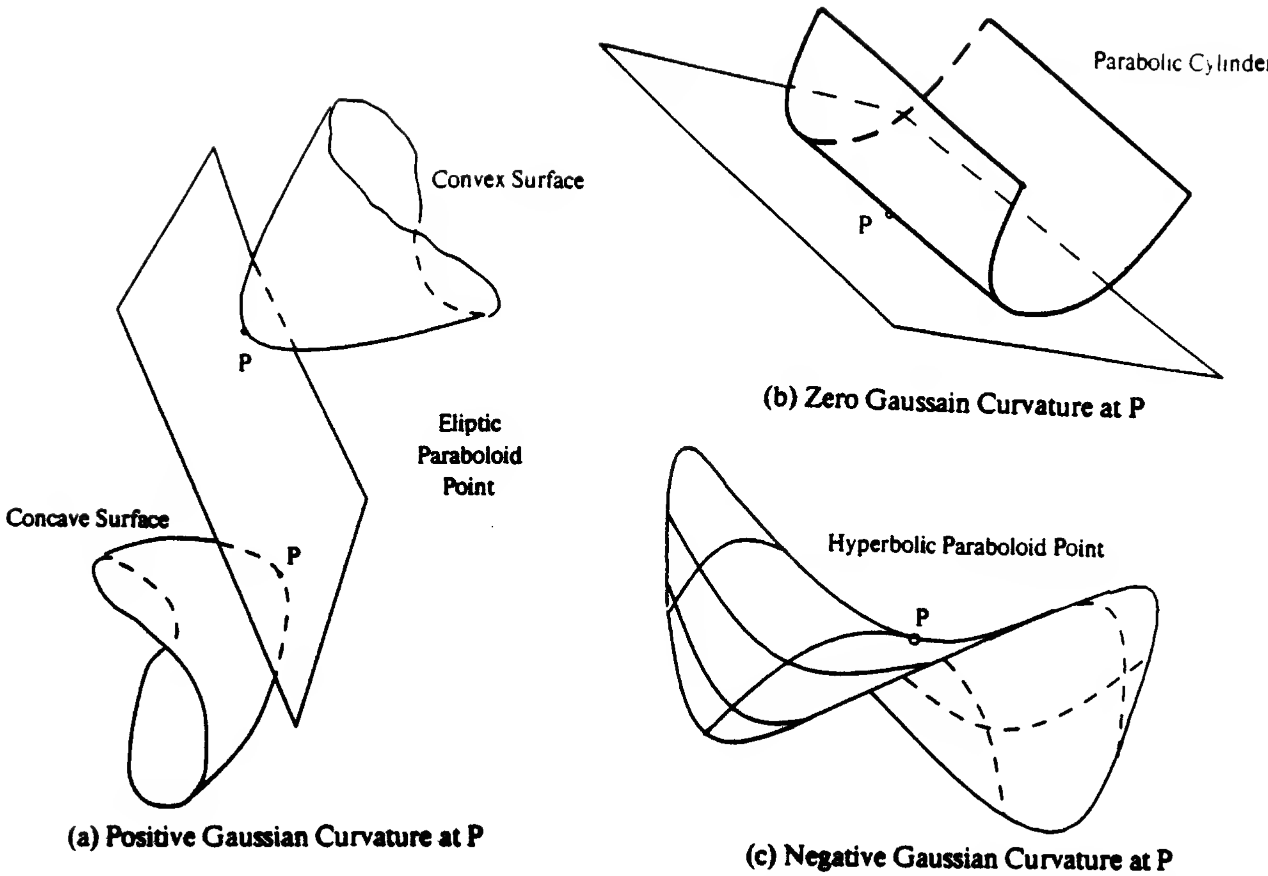

Figure 2: Examples of Surfaces with Positive, Zero, and Negative Gaussian Curvature

Finally consider the case when the surface has a negative gaussian curvature, that is $|H_{ij}| < 0$ at O (see Figure 2c). The point of fixation O is referred to as a *hyperbolic paraboloid* or a *saddle* point. In this case, both solutions of θ are real, and given by:

$$\tan\theta = \frac{-d_{12} \pm \sqrt{d_{12}^2 - d_{11}d_{22}}}{d_{11}}.$$

Appendix 2:

In this appendix, we present the results of section 2 in the vector form for those readers that are not familiar with the tensor notation.

2.1. Perspective Projection

Let $\mathbf{x}$ denote the perspective projection of the point $\mathbf{X}$ onto the image plane (plane $z = 1$). Then:

$$\mathbf{x} = \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}} \mathbf{X}.$$

2.2. Rigid Body Motion

Let $\mathbf{t}$ and ω denote the translational and rotational velocities of the observer relative to the scene, then:

$$\frac{d\mathbf{X}}{dt} = -\boldsymbol{\omega} \times \mathbf{X} - \mathbf{t}.$$

2.3. Image Motion Field and Optical Flow

The projection of the 3-D velocity field, induced by a rigid body motion, onto the image is known as the *image motion field*, and is given by:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt} \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}} \mathbf{X}.$$

For convenience, we introduce the notation $\mathbf{x}_t$ and $\mathbf{X}_t$ for the time derivatives of $\mathbf{x}$ and $\mathbf{X}$, respectively. We then have

$$\mathbf{x}_t = \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}} \mathbf{X}_t - \frac{1}{(\mathbf{X} \cdot \hat{\mathbf{z}})^2} (\mathbf{X}_t \cdot \hat{\mathbf{z}}) \mathbf{X},$$

which can also be written in the compact form:

$$\mathbf{x}_t = \frac{1}{(\mathbf{X} \cdot \hat{\mathbf{z}})^2} \big(\hat{\mathbf{z}} \times (\mathbf{X}_t \times \mathbf{X})\big),$$

since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. Note that $\mathbf{X} = (\mathbf{X} \cdot \hat{\mathbf{z}})\mathbf{x}$, and therefore,

$$\mathbf{x}_t = \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}} \big(\hat{\mathbf{z}} \times (\mathbf{X}_t \times \mathbf{x})\big).$$

Finally, substituting for $\mathbf{X}_t$, we obtain:

$$\mathbf{x}_t = -\Big(\hat{\mathbf{z}} \times \big(\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega} - \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}} \mathbf{t})\big)\Big).$$

2.4. Brightness Change Constraint Equation

The brightness of the image of a particular patch of a surface depends on many factors. It may for example vary with the orientation of the patch relative to the viewer or the

light source. In many cases, however, it remains at least approximately constant as the surface moves in the environment. If we assume that the image brightness of a patch remains constant, we have:

$$\frac{dE}{dt} = 0,$$

or

$$\frac{\partial E}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial E}{\partial t} = 0,$$

is the image brightness gradient. It is convenient to use the notation $E_{\mathbf{x}}$ for this quantity and E_t for the time derivative of the brightness. Then we can write the brightness change equation in the simple form

$$E_{\mathbf{x}} \cdot \mathbf{x}_t + E_t = 0.$$

Substituting for $\mathbf{x}_t$ we get

$$E_t - E_{\mathbf{x}} \cdot \left(\hat{\mathbf{z}} \times \left(\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega} - \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}}\mathbf{t})\right)\right) = 0.$$

If can be eaasily verified that if we define $\mathbf{s} = (E_{\mathbf{x}} \times \hat{\mathbf{z}}) \times \mathbf{x}$, and $\mathbf{v} = -\mathbf{s} \times \mathbf{x}$, then the above equation can be written in the simple form:

$$E_t + \mathbf{v} \cdot \boldsymbol{\omega} + \frac{1}{\mathbf{X} \cdot \hat{\mathbf{z}}}\mathbf{s} \cdot \mathbf{t} = 0.$$

This is the ***brightness change constraint equation*** in the case of rigid body motion.

Appendix 3:

We defined $d = 1/Z$. Therefore:

$$\frac{\partial d}{\partial x_i} = -\frac{1}{Z^2}\frac{\partial Z}{\partial x_i}, \qquad \frac{\partial^2 d}{\partial x_i \partial x_j} = -\frac{1}{Z^2}\frac{\partial^2 Z}{\partial x_i \partial x_j} + \frac{2}{Z^3}\left(\frac{\partial Z}{\partial x_i}\right)\left(\frac{\partial Z}{\partial x_j}\right).$$

Evaluating these expressions at the origin:

$$\left(\frac{\partial d}{\partial x_i}\right)_0 = -\frac{1}{Z_0^2}\left(\frac{\partial Z}{\partial x_i}\right)_0, \qquad \left(\frac{\partial^2 d}{\partial x_i \partial x_j}\right)_0 = -\frac{1}{Z_0^2}\left(\frac{\partial^2 Z}{\partial x_i \partial x_j}\right)_0 + \frac{2}{Z_0^3}\left(\frac{\partial Z}{\partial x_i}\right)_0\left(\frac{\partial Z}{\partial x_j}\right)_0.$$

We now use the perspective projection equation $X_i = x_i Z$:

$$\frac{\partial Z}{\partial x_i} = \frac{\partial Z}{\partial X_m}\frac{\partial X_m}{\partial x_i} = \frac{\partial Z}{\partial X_m}\left(\delta_{im} Z + x_m \frac{\partial Z}{\partial x_i}\right).$$

Therefore:

$$\frac{\partial Z}{\partial x_i} = Z\frac{\partial Z}{\partial X_i}\left(1 - x_m \frac{\partial Z}{\partial X_m}\right)^{-1}.$$

Now:

$$\frac{\partial^2 Z}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}(\frac{\partial Z}{\partial X_m})(\delta_{im} Z + x_m \frac{\partial Z}{\partial x_i}) + \frac{\partial Z}{\partial X_m}(\delta_{im}\frac{\partial Z}{\partial x_j} + \delta_{jm}\frac{\partial Z}{\partial x_i} + x_m \frac{\partial}{\partial x_j}(\frac{\partial Z}{\partial x_i})),$$

or,

$$\frac{\partial^2 Z}{\partial x_i \partial x_j} = \frac{\partial}{\partial X_n}(\frac{\partial Z}{\partial X_m})\frac{\partial X_n}{\partial x_j}(\delta_{im} Z + x_m \frac{\partial Z}{\partial x_i}) + \frac{\partial Z}{\partial X_m}(\delta_{im}\frac{\partial Z}{\partial x_j} + \delta_{jm}\frac{\partial Z}{\partial x_i} + x_m \frac{\partial^2 Z}{\partial x_i \partial x_j}).$$

This simplifies to:

$$\frac{\partial^2 Z}{\partial x_i \partial x_j} = \frac{\partial^2 Z}{\partial X_m \partial X_n}(\delta_{im} Z + x_m \frac{\partial Z}{\partial x_i})(\delta_{jn} Z + x_n \frac{\partial Z}{\partial x_j}) + \frac{\partial Z}{\partial X_m}(\delta_{im}\frac{\partial Z}{\partial x_j} + \delta_{jm}\frac{\partial Z}{\partial x_i} + x_m \frac{\partial^2 Z}{\partial x_i x_j}).$$

After substituting for $\partial Z/\partial x_i$, we arrive at:

$$\frac{\partial^2 Z}{\partial x_i \partial x_j} = \frac{\partial^2 Z}{\partial X_m \partial X_n}\tilde{Z}_{im}\tilde{Z}_{jn} + \frac{\partial Z}{\partial X_m} + \left[Z(1 - x_k \frac{\partial Z}{\partial X_k})^{-1}(\delta_{im}\frac{\partial Z}{\partial X_j} + \delta_{jm}\frac{\partial Z}{\partial X_i}) + x_m \frac{\partial^2 Z}{\partial x_i x_j}\right],$$

where,

$$\tilde{Z}_{im} = \delta_{im} Z + x_m Z \frac{\partial Z}{\partial X_i}(1 - x_k \frac{\partial Z}{\partial X_k})^{-1}.$$

Solving for $\partial^2 Z/\partial x_i x_j$, we obtain:

$$\frac{\partial^2 Z}{\partial x_i \partial x_j} = (1 - x_r \frac{\partial Z}{\partial X_r})^{-1}\tilde{Z}_{im}\tilde{Z}_{jn}\frac{\partial^2 Z}{\partial X_m \partial X_n} + Z(1 - x_k \frac{\partial Z}{\partial X_k})^{-2}(\delta_{im}\frac{\partial Z}{\partial X_j} + \delta_{jm}\frac{\partial Z}{\partial X_i})\frac{\partial Z}{\partial X_m}.$$

Evaluating $(\partial Z/\partial x_i)_0$ and $(\partial^2 Z/\partial x_i \partial x_j)_0$ at the origin results in:

$$(\frac{\partial Z}{\partial x_i})_0 = Z_0 \frac{\partial Z}{\partial X_i}_0, \qquad (\frac{\partial^2 Z}{\partial x_i x_j})_0 = Z_0^2(\frac{\partial^2 Z}{\partial X_i \partial X_j})_0 + 2Z_0(\frac{\partial Z}{\partial X_i})_0(\frac{\partial Z}{\partial X_j})_0.$$

Finally, substituting for $(\partial Z/\partial x_i)_0$ and $(\partial^2 Z/\partial x_i \partial x_j)_0$ into the equations for $(\partial d/\partial x_i)_0$ and $(\partial^2 d/\partial x_i \partial x_j)_0$ we arrive at:

$$(\frac{\partial d}{\partial x_i})_0 = -\frac{1}{Z_0}(\frac{\partial Z}{\partial X_i})_0, \qquad (\frac{\partial^2 d}{\partial x_i \partial x_j})_0 = -(\frac{\partial^2 Z}{\partial X_i \partial X_j})_0.$$

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